JIFM JOURNAL OF INSURANCE AND FINANCIAL MANAGEMENT

Flash Crashes: The Role of Information Processing Based Subordination and the Cauchy Distribution in Market Instability

Edgar Parker [a,*]

[a] *New York Life Insurance Company, United States*

ARTICLE INFO

JEL Classification:
G10
G12
G14
G17
D80

Keywords:
Subordination
Cauchy Distribution
Flash Crash
High Frequency Trading
Brownian Motion
Information Theory

ABSTRACT

While a wide variety of hypotheses have been offered to explain the anomalous market phenomena known as a "Flash Crash", there is as of yet no consensus among financial experts as to the sources of these sudden market collapses. In contrast to the behavior expected from standard financial theory, both the equity and bond markets have been thrown into freefall in the absence of any significant news event. The author posits that a combination of probability and information theory, and diffusion dynamics offers a relatively simple explanation of the causes of some of these dramatic events. This new avenue of research also suggests new policies or measures to lower the probability of occurrence and to mitigate the effects of these extreme events.

Journal of Insurance and Financial Management

*Corresponding Author:
edgar_parker@newyorklife.com

The author would like to thank Brian Jawin, Brian Kwei, Todd Taylor, and Gary Ng of New York Life, and Dale Hanley and Jennifer Ni of NYLIM for their suggestions and encouragement.

1. Introduction

Two of the key variables studied in this paper that affect the behavior of the markets are the information arrival (CC_A) and processing rates (CC_L) of the market (and by extension the market participants). As will be shown the value of the ratio of these rates (CC_A/CC_L) can determine different regimes of normal and “anomalous” behaviors for security returns. As this ratio evolves over a continuum of values, security returns can be expected to go through phase transitions between different types of behavior. These dramatic phase transitions can occur even when the underlying information generation mechanism is unchanged. Additionally when the information arrival and processing rates are assumed to fluctuate independently and normally, the resulting ratio (CC_A/CC_L) is shown to be Cauchy distributed and thus fat tailed.

This line of research actually suggests significantly different remedies to market instability compared to those currently utilized such as so called “circuit breakers”. The level and stability of the information processing rate CC_L of the market (and market participants) turns out to be the most important variable in the model. Policies which increase this level and minimize the variance of the information processing rate will reduce the probability of occurrence and the ultimate severity of anomalous market fluctuations.

1.1 Non-normality, Subordination, and the Limits of Computation

One obvious and well documented feature of securities markets is the observed non-normality of market returns. The concept of trading time or subordinated Brownian motion as pioneered by (Mandelbrot & Taylor, 1967), and Clark (1973) provides a method of retrieving the normality assumption. The traditional calendar clock is replaced with stochastic time due to the random arrival of information at the market. Market activity in terms of the number and/or size of trades is often used as a proxy of the information arrival rate. After taking into account this stochastic changing of time or “time deformation” returns have been demonstrated to be approximately normally distributed for most return data.

1.2 Failure of the Subordination Hypothesis for Large Price Movements

The time deformation or subordination methods have been extensively studied and provide a useful explanation and remedy for a large bulk of the observed market non-normality. However, there is evidence that subordination fails to completely explain extreme price movements. Silva (2005) determined that subordination only effectively explains the center ($\approx$ 85%) of price movements, and that the subordination hypothesis is rejected for returns

that reach the 2 standard deviation level and above. While (Farmer, Gillemot, Lillo, Mike, & Sen, 2004) provides evidence that subordination does not explain large price movements in the London Stock Exchange.

This author argues that prior time deformation methods are missing an additional important factor which may affect the subordination process. The missing factor is the information processing ability of the market (and market participants) relative to the randomly arriving information. The decisive role of the information processing rate and its variance in the behavior of market returns is the subject the following analysis.

In the following section an alternative formulation of the standard subordination process is derived and its relation to the typical trade number and volume type time deformation process is presented. Afterwards the author shows how this alternative structure naturally generates different behaviors including the increased likelihood of large price movements. Finally the remedies suggested by this perspective to reduce the severity and probability of occurrence of these extreme events are discussed.

2. Model Subordination by the Information Processing Clock

Extending the presentation style of (Huth & Abergel, 2012) this author introduces the new concept of the relative excess (or unprocessed) information. Information that is not immediately processed is not somehow rendered irrelevant. Unprocessed information in a current time period will have to be processed at some future time period in order for no arbitrage arguments to hold.

Ross (1989) found that changes in the resolution of uncertainty (or the arrival time of information) will change current prices only if they alter the cash flows or equivalently alter the average standard deviation of the price process. However future prices will certainly be affected by future information or information whose arrival is postponed into the future. Prices will respond at the time that the information is eventually processed.

Intermediate asset prices can diverge from those implied by the supporting information series if the information processing rate falls behind the information arrival rate. Although prices may later converge to those indicated by the full information stream, intermediate prices may deviate wildly from those rationally implied by concurrent information arrival and processing. A normally distributed information arrival process may ultimately result in a highly volatile and non-normal price return distribution.

Additionally unprocessed information may build up until some point in the future where processing expands and/or current information generation drops to a level where it can be appropriately analyzed. The resolution of this backlogged information would lead to apparent clustering of price changes. This clustering of processed information would occur in spite of a hypothetically Gaussian informal arrival mechanism.

First we present the typical subordination in the style of (Huth & Abergel, 2012):

$$F_i = \frac{P_i}{P_{i-1}}; performance\ of\ asset\ between\ trades\ (i-1) and\ i$$

$$Total\ Variation = \frac{P_N}{P_0} = \prod_{i=1}^{N} F_i$$

$$R_{\frac{CCa}{CCl}} = \ln\left(\frac{P_N}{P_0}\right) = \sum_{i}^{N} \ln(F_i)$$

$$\lim_{N\to\infty} Var\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N} X_1\right) = Var(X_1) + 2\sum_{k=1}^{+\infty} Cov(X_1, X_{1+k})$$

$$\frac{R_N}{\sqrt{N}} = N\ (0, \sigma^2)$$

$$R_N \sim N\ (0, N\sigma^2)$$

A similar result can be obtained for volume, see (Huth & Abergel, 2012):

$$R_V \sim N\ \left(0, \frac{V\sigma^2}{E(V_1)}\right)$$

An extension is made of the above logic with the addition of the new concept of excess (or unprocessed) information in the ith information set. The key result of additional return variance beyond that of traditional subordination is caused by this excess information:

$$F_i = \frac{P_i}{P_{i-1}} = \frac{I_i}{I_{i-1}}; by\ \ Ross\ (1989)$$

$$Let\ Cl_i = information\ processed\ in\ trade\ i$$

$$Ie_i = \max(I_i, I_i/Cl_i) = excess\ information\ in\ trade\ i$$

If all information is processed at i, $I_i = Cl_i\ \ then\ Ie_i = I_i$

$$otherwise\ Ie_i = I_i/Cl_i$$

$$Let\ CCa_i = \frac{I_i}{I_{i-1}};\ CCl_i = \frac{Cl_i}{Cl_{i-1}}$$

$$\frac{CCa_i}{CCl_i}$$

$$\frac{I_{ei}}{Ie_{i-1}} = \frac{I_i/Cl_i}{I_{i-1}/Cl_{i-1}} = \frac{I_i/I_{i-1}}{Cl_i/Cl_{i-1}} = \frac{CCa_i}{CCl_i}; \frac{CCa_i}{CCl_i} = \min\left(1, \frac{CCa_i}{CCl_i}\right)$$

There is a greater amount of information per trade than has been analyzed by the market in N trades.

$$N_{total} = \text{total information content of N trades} = N * excess\ information\ rate$$

$$N_{total} = N * \frac{CCa}{CCl}$$

$$F_i = \frac{I_{ei}}{I_{ei-1}} = \frac{I_{\left(N*\frac{CCa}{CCl}\right)}}{I_{N*\frac{CCa}{CCl}-1}}$$

$$\frac{I_{\left(N*\frac{CCa}{CCl}\right)}}{I_0} = \prod_{i=1}^{N} F_i$$

$$R_{\frac{CCa}{CCl}} = \ln\left(\frac{I_{\left(N*\frac{CCa}{CCl}\right)}}{I_0}\right) = \sum_{i}^{N*\frac{CCa}{CCl}} \ln(F_i)$$

$$\lim_{N*\frac{CCa}{CCl}\to\infty} Var\left(\frac{1}{\sqrt{N*\frac{CCa}{CCl}}} \sum_{i=1}^{N*\frac{CCa}{CCl}} X_1\right) = Var(X_1) + 2\sum_{k=1}^{+\infty} Cov(X_1, X_{1+k})$$

$$\frac{R_{N*\frac{CCa}{CCl}}}{\sqrt{N*\frac{CCa}{CCl}}} = N\ (0, \sigma^2)$$

$$R_{N*\frac{CCa}{CCl}} = N\ \left(0, N*\frac{CCa}{CCl}\sigma^2\right)$$

Or similarly for volume:

$$R_{V*\frac{CCa}{CCl}} \sim N\ \left(0, \frac{V*\frac{CCa}{CCl}\sigma^2}{E(V_1)}\right)$$

Now the total variance is dependent on the amount of information in a similar fashion to trading time subordination. However the total variance is also inversely related to the speed of information processing in the market.

If all of the information generated is processed ($CCa = CCl$); then the preceding reduces to the standard trading time (Number of trades) relationship:

$$All\ information\ processed;\ \frac{CCa}{CCl} = 1$$

$$R_{N*\frac{CCa}{CCl}} = R_N = N\ (0, N\sigma^2)$$

In summary when the market processes all information associated with each trade the typical trade size subordination holds. However when there is excess or unprocessed information (CC_A/CC_L >1), then the return variance will be greater than that implied by the typical subordination relationship. This is the source of the failure of the subordination hypothesis reported by Silva (2005) for approximately 15% of returns, and for (Farmer et al., 2004) for large price movements.

$$If\ \frac{CCa}{CCl} \leq 1; Variance = N * \frac{CCa}{CCl}\sigma^2 = N * (1) * \sigma^2 = N\sigma^2$$

$$If\ \frac{CCa}{CCl} > 1; Variance = N * \frac{CCa}{CCl}\sigma^2 > N\sigma^2$$

3. Analysis

3.1 Phase Transitions (The Level of CC_L)

Utilized in conjunction with the new variance relationship derived above, diffusion theory provides a revealing perspective to analyze the transitions between the “normal” and anomalous behavior of security returns. This framework shows how changes in information processing can lead price movements to experience transformations in their diffusive character. These transitions lead to different periods of Gaussian and non-Gaussian behavior even under the assumption of a stable and Gaussian information arrival process. These different periods can be viewed as phase transitions in behavior as distinct as the transition of water to steam and ice.

(Plerou, Gopikrishnan, Amaral, Gabaix, & Stanley, 2000) found that stock price movements are equivalent to a complex variant of classical diffusion. Agent actions and interactions were shown to generate non normal price behavior even when the underlying information

process is Gaussian by (Lux & Marchesi, 1999). (Parker, 2013, 2015) used the various states of diffusion as described by Klages (2010) and information theory to describe regimes of price return behavior.

The author suggests that differing diffusive states of the price process can be generated if the information processing rate of the market equals, exceeds, or is less than the information reception rate from the outside environment. The possible regimes are the normal, subdiffusive, and superdiffusive states. The normal or Gaussian diffusive state as identified by Bachelier (1900) in finance and Einstein (1905) in the Brownian motion of colloidal particles suspended in a solution is the diffusion type typically assumed in most financial theory.

When excess information is generated the diffusive nature of the original information series can be amplified or pushed to greater diffusion as reflected in the final price series. In the context of security returns when $CC_A/CC_L > 1$, the lowest achievable diffusion of the original information series is moved upward towards greater diffusion in terms of the resultant price series. An information series that has a normal Gaussian diffusion may result in a price series with a higher adjusted diffusion and ultimately be in a super diffusive state. The result would be extreme movements in the return distribution despite the arrival of normally distributed information.

States of Diffusion

$$\frac{CCa}{CCl} \leq 1; Diffusion\ increased\ equivalent\ to\ typical\ subordination$$

$$N * \frac{CCa}{CCl}\sigma^2 = N\sigma^2$$

$$\frac{CCa}{CCl} > 1; greater\ push\ to\ superdifussion\ than\ typical\ subordination$$

$$N * \frac{CCa}{CCl}\sigma^2 > N\sigma^2$$

As the ratio CC_A/CC_L varies over time, an information series that is subdiffusive and non Gaussian may be moved across the entire spectrum of behavior from sub to normal to super diffusion as measured by the price series. This evolution in price behavior can occur even if the information arrival process experienced no such change. From this vantage point dramatic changes in market behavior are a natural outcome of changing information creation and processing rates as represented in the relationship between them in the ratio CC_A/CC_L.

Additionally, measures that increase the level of CC_L will reduce the occurrence of extreme price changes.

3.2 Information Processing and the Cauchy Distribution (The Variance of CC_L)

In this section the analysis will be extended by assuming that both CC_A and CC_L are generated as normal random variables. Under these assumptions the information arrival and processing rates of the markets (and agents) fluctuate randomly. The simple model built upon these assumptions will illustrate the importance of the variance CC_L in the behavior of the price return process.

(Bohacek & Rozovskii 2004) studied the travel times of information packets over the internet. Queuing delay resulted in a random time varying component to their model (Footnote: Interestingly they utilized models originally developed to model short rates in finance, further illustrating the linkage between finance, information theory, and information processing). Similarly the different components of an agent's information processing system can be assumed to experience similar queuing delays and the resulting randomly evolving processing rates $CC_L(t)$.

$$CCa_t = CCa_0 + \sigma Z_1;\ Z_1 \sim N(0,1)$$

$$CCl_t = CCl_0 + \sigma Z_2;\ Z_2 \sim N(0,1)$$

Let $W_t = \frac{CCa_t}{CCl_t}$

It can be shown W_t is a random variable with a Cauchy distribution which is one of the few stable but non-normal distributions. The Levy, the Normal, and the Cauchy are the only stable distributions with analytical probability density functions. Mandelbrot (2006) among others favored non-normal distributions (such the Levy and Cauchy distributions) as better approximations of stock and commodity price behavior. Specifically he studied "Stable Pareto Distributions" which similar to Cauchy distributions result in fatter tails when compared to the normal distributions. Cauchy distributions actually have nonfinite (or undefined) means and variances. The fatter tails and undefined variances cause extreme events to occur much more frequently compared to a process modeled with the normal distribution.

Below we demonstrate that $W_t = \frac{CCa_t}{CCl_t}$ has the standard Cauchy distribution:

$$Z_1 = \frac{CCa_t - CCa_0}{\sigma}$$

$$Z_2 = \frac{CCl_t - CCl_0}{\sigma}$$

$$f_{Z_1}(Z_1) = \frac{e^{Z_1^2/2}}{\sqrt{2\pi}};\ -\infty < Z_1 < \infty$$

$$f_{Z_2}(Z_1) = \frac{e^{Z_2^2/2}}{\sqrt{2\pi}};\ -\infty < Z_2 < \infty$$

$$f_{Z_1,Z_2}(Z_1, Z_2) = \frac{e^{-(Z_1^2+Z_2^2)/2}}{2\pi}; -\infty < Z_1 < \infty, -\infty < Z_2 < \infty$$

$$Y_1 = h_1(Z_1, Z_2) = \frac{Z_1}{Z_2}$$

$$Y_2 = h_2(Z_1, Z_2) = Z_2$$

$$Z_1 = h_1{}^{-1}(Y_1, Y_2) = Y_1 Y_2 \ and\ Z_2 = h_2{}^{-1}(Y_1, Y_2) = Y_2$$

$$J = \begin{vmatrix} Y_2 & Y_1 \\ 0 & 1 \end{vmatrix} = Y_2$$

$$f_{Y_1,Y_2}(y_1, y_2) = f_{Z_1,Z_2}\left(h_1{}^{-1}(y_1, y_2), h_1{}^{-1}(y_1, y_2)\right)|J|$$

$$= \frac{e^{-\left(y_1^2 y_2^2 + y_2^2\right)/2}}{2\pi}|y_2|$$

$$= \frac{y_2 e^{-y_2^2(y_1^2+1)/2}}{2\pi}$$

$$= \frac{1}{\pi(y_1^2 + 1)};\ -\infty < y_1 < \infty$$

This is the probability density function of a standard Cauchy random variable.

Substituting the new Cauchy distributed $W_t = \frac{CCa_t}{CCl_t}$ into the main equation is illustrated below.

$$N * \frac{CCa}{CCl}\sigma^2 = NW_t\sigma^2$$

Now the values of the critical ratio ($\frac{CCa_t}{CCl_t} = W_t$) are driven by a process where extreme values are to be expected more often than under normal assumptions.

This analysis suggests a potential new means of stabilizing market behavior. Specifically the variance of CC_L could be reduced so that it is more dominated by a high level of CCl_0 and less by the variance of the information processing structure. As CC_L approaches a more constant level with near zero variance W_t would essentially be transformed into a normally distributed random variable whose behavior would be dominated by CCa. A more reliable and stable information processing structure would accomplish this as seen below:

Increase CCl_0 and minimize the variance of Z_2 in such a way so that this stable term dominates the relationship:

$$CCl_t = CCl_0 + \sigma Z_2;\ \ Z_2 \sim N(0,1); becomes$$

$$CCl_t \approx CCl_0$$

This transforms $W_t = \frac{CCa_t}{CCl_t}$ into a form determined only by the normally distributed CCa.

$$From\ Cauchy\ Distributed\ W_t = \frac{CCa_t}{CCl_0}$$

$$To\ Normally\ distributed\ W_t = \frac{CCa_t}{CCl_0};$$

$$where\ CCa_t = CCa_0 + \sigma Z_1;\ \ Z_1 \sim N(0,1)$$

$$W_t = (\frac{1}{CCl_0})CCa_0 + \sigma Z_1$$

(Note: A key difference in this discussion from some of the usual modeling debates is I am not assuming whether the original structure of returns is Gaussian or Cauchy, or any other non-normal form. I am arguing that the trading system can be designed and modified such that return behavior more closely approximates a Gaussian versus a Cauchy type distribution.)

In Tables 1 and 2 below are two examples each with 50,000 random draws where

$X\ and\ Y \sim N(0,1)$.

Table 1
50,000 Random Draws of X and Y
X and Y~N(0,1)

Variable	Mean	Variance
X	-0.005	0.992
Y	0.005	0.998
Z = X/Y	0.902	71,149.79

Similar draws in Table 2 illustrate the dramatic change in the Cauchy distributed Z.

Table 2
50,000 Random Draws of X and Y
X and Y~N(0,1)

Variable	Mean	Variance
X	0.000	0.992
Y	-0.001	0.999
Z = X/Y	53.975	144,400,419.61

To further illustrate the difference in behavior between Z and its constituents a graph of the first 50 values of a typical run is presented below in Figure 1. As seen in the graph, despite being composed of X and Y which are normally distributed, clearly Z has dramatically different behavior. The Z distribution is populated with extreme values of greater magnitude and frequency.

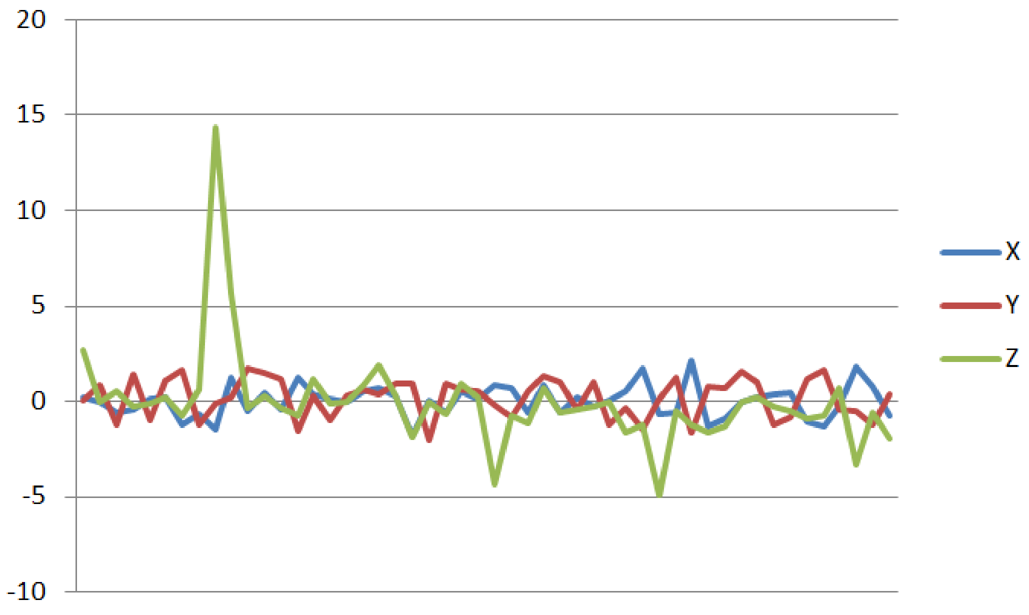

Figure 1
First 50 Values of X, Y, and Z

Next the effect of reducing the variance of Y (representing a reduction in the variance of CC_L) on the Z distribution is presented. Similar to the above example X and Y are IID with the mean and variance described below in Table 3:

$$X \; and \; Y \sim N(1,1)$$

Here we are assuming the average information generation and processing rates are nonzero. Table 3 below illustrates that the mean and variance of Z is still nonfinite and does not converge with added terms or repeated runs:

Table 3
50,000 Random Draws of X and Y
X and Y~N(1,1)

Variable	Mean	Variance
X	0.989	1.000
Y	0.999	1.003
Z = X/Y	1.340	298,138.96

However if the variance of Y (representing a reduction in the variance of CC_L) is dramatically reduced then the mean and variance of Z begins to assume the behavior of X. The magnitude and frequency of extreme values are now reduced and $X \sim N(1,1) \; and \; Y \sim N(1,0.01)$ in Table 4 below:

Table 4
50,000 Random Draws of X and Y
X~N(1,1) and Y~N(1,0.01)

Variable	Mean	Variance
X	0.997	1.002
Y	1.001	0.010
Z = X/Y	1.006	1.04

4. Conclusions

As shown in this paper dramatic market moves such as Flash Crashes are a natural outcome of the changing information creation and processing rates, and the relationship between them as captured in the ratio CC_A/CC_L. Additionally these relationships suggest that current measures used to mitigate the effects of dramatic market declines may actually exacerbate the situation.

Instead of speeding the market’s processing of information (including perhaps erroneous information or noise which nonetheless must still be processed) circuit breaker type interventions tend to result in the reverse effect. Circuit breakers halt a trader’s ability to act on market information, and also increase the variability of the rate of such processing. However this trading halt neither eliminates existing information nor does it not stop the generation of new market information. Continued information generation can lead to further increases in the critical ratio CC_A/CC_L while the increased variability of CC_L can push the dynamics into a more extreme regime. This analysis leads to the unfortunate realization that circuit breakers can actually precipitate the very behavior they are designed to curb.

The relationships brought to light by this new perspective suggest that measures that increase the speed and decrease of the variability of information processing would be preferable to trading halts. Additionally measures should be taken that encourage relatively faster information processing participants such as HFT’s to remain active market participants during crises. Anticipated trading halts turn the markets into a panicked race for the exit when volatility becomes extreme. This happens tragically at the very time that reliable high speed information processing is needed.

5. References

Bachelier, L. (1900). The theory of speculation, *Annales Scientifiques de l'E´cole Normale Supe´rieure Se´r. 3*(17), 21–86.

Bohacek, S., & Rozovskii, B. A. (2004). Diffusion model of roundtrip time. *Computational Statistics and Data Analysis 45*, 25-50.

Christian, S. (2005). Applications of physics to finance and economics: Returns, trading activity and income, Phd Thesis, Department of Physics, University of Maryland.

Clark, P. K. (1973). A subordinated stochastic process model with finite variance for speculative prices. *Econometrica 41*, 135-155.

Einstein, A. (1905). On the movement of small particles suspended in a stationary liquid demanded by the kinetic molecular theory of heat, *Annalen. der Physik, 4. Folge, 17*, 549-560.

Farmer, J. D., Gillemot, L., Lillo, F., Mike, S., & Sen, A. (2004). What really causes large price changes, *Quantitative Finance 4*(4), 383-397.

Huth, N., & Abergel, F. (2012). The times change: Multivariate subordination. Empirical facts, *Quantitative Finance*, *12*(1) 1-10.

Klages, R. (2010). Reviews of Nonlinear Dynamics and Complexity Volume 3, Heinz Georg Schuster (Ed.). Weinheim, Germany: Wiley-VCH Verlag GmbH & Co. KGaA.

Lux, T., & Marchesi, M. (1999). Scaling and criticality in a stochastic multi-agent model of a financial market, *Nature*, *397*, 498–500.

Mandelbrot, B. (2006). The misbehavior of markets: A fractal view of risk, ruin, and reward. New York, NY: Basic Books.

Mandelbrot, B., & Taylor, H. M. (1967). On the distribution of stock price differences, *Operations Research Vol. 15*(6) 1057-1062.

Parker, E. (2013). Efficient markets meet the shannon limit (The shannon limit, relative channel capacity, and price uncertainty). [Kindle DX version]. Retrieved from Amazon.com or http://dx.doi.org/10.2139/ssrn.2516557 .

Parker, E. (2015). Entropy production and technological progress: The yin and yang of economics and finance. [Kindle DX version]. Retrieved from Amazon.com or http://dx.doi.org/10.2139/ssrn.2684841 .

Plerou, V., Gopikrishnan, P., Amaral, L. A. N., Gabaix, X., & Stanley, H. E. (2000). Economic fluctuations and anomalous diffusion. *Physical Review E 62*(3), 3023-3026.

Ross, S. A. (1989). The no-arbitrage martingale approach to timing and resolution irrelevancy. *The Journal of Finance, 44*, 1-17.